Department of Complexity Science and Engineering,
Graduate School of Frontier Scienses, The University of Tokyo
Graduate School Entrance Examination Problems for Academic Year 2021
Examination duration: 30 minutes
(Sample 1)

Answer the following questions. All the constants, variables, and functions given below are real.

(Q.1) Consider the surface $z = ax^2+y^2+2xy-2x$ on xyz-space in the Cartesian coordinate system,
where $a \neq 0$.

(i) Let $z = rx+py+s$ be the tangent plane to the surface at $(x, y, z) = (1, 0, a-2)$. Obtain r, p, and s.

(ii) Find the condition on a such that the tangent plane obtained in (i) has an intersection with x-axis.

(iii) Suppose that the distance between the origin and the plane satisfying the condition (ii) is $\frac{1}{\sqrt{5}}$. Obtain a.

(Q.2) Consider matrices and vectors given by

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \boldsymbol{v} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \ A_t = D + t\boldsymbol{v}\boldsymbol{v}^\top, \ \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where t is a positive real number and $\top$ indicates the tranpose.

(i) Express the sum of all elements of A_t in terms of t.

(ii) Let b_{ij} be the (i, j)-th element of A_t^{-1}. Express b_{11} and b_{33} in terms of t.

(iii) Assume that $\boldsymbol{x}$ is a constant vector independent of t. Obtain the necessary and sufficient condition for $\boldsymbol{x}$ so that $\lim_{t\to\infty} \boldsymbol{x}^\top A_t^{-1} \boldsymbol{x} = 0$.

(Q.3) Let $f(y)$ be a function $f(y) = y - y^2$.

(i) Obtain the constants A and B for the partial fraction decomposition $\dfrac{1}{f(y)} = \dfrac{A}{y} + \dfrac{B}{1-y}$ $(y \neq 0, 1)$.

(ii) Obtain the general solution to the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(y)$. Use C as an arbitrary constant.

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(Sample 2)

Answer the following questions. All the constants, variables, and functions given below are real.

(Q.1) Consider a line $\ell : y = ax + b$ on xy-plane, where a and b are real numbers.

(i) Suppose that the distance between the point $(0, q)$ and the line is $2/\sqrt{a^2+1}$, where q is a real number. Find the value of q.

(ii) Suppose that ℓ is the line that fits the three points $(1,0), (0,-1)$, and $(2,p)$ obtained by the least squares method, where p is a real number. Express a and b in terms of p.

(iii) Find the point through which the approximate line obtained in (ii) passes for any p.

(Q.2) Consider matrix A with its inverse A^{-1} and vectors $\boldsymbol{a}$ and $\boldsymbol{x}$ in the Cartesian coordinate system given by

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad A^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & 3 \\ 6 & -2 & 1 \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Let plane S be the set of points $\boldsymbol{x}$ satisfying $\boldsymbol{a}^\top \boldsymbol{x} = 1$, where $\top$ indicates the transpose.

(i) Let $S' = \{\boldsymbol{x}' = A\boldsymbol{x} : \boldsymbol{x} \in S\}$ be the image of S through linear mapping $f(\boldsymbol{x}) = A\boldsymbol{x}$.
Obtain $\boldsymbol{b} = (b_1, b_2, b_3)^\top$ such that the plane S' is expressed by the equation $\boldsymbol{b}^\top \boldsymbol{x}' = 1$.

(ii) Obtain the coordinates of the foot of the perpendicular from the origin to S'.

(iii) For $c > 0$, let T be the three-dimensional ellipsoid described by the set of points $\boldsymbol{x}$ satisfying $\boldsymbol{x}^\top A^\top A \boldsymbol{x} = c$. Obtain c such that T is tangent to S.

(Q.3) Let $f(y)$ be a function $f(y) = 1 - \frac{1}{4}y^2$.

(i) Obtain the constants A and B for the partial fraction decomposition $\dfrac{1}{f(y)} = \dfrac{A}{1+\frac{y}{2}} + \dfrac{B}{1-\frac{y}{2}}$ $(y \neq \pm 2)$.

(ii) Obtain the general solution to the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(y)$. Use C as an arbitrary constant.