Department of Complexity Science and Engineering
Graduate School of Frontier Sciences
The University of Tokyo

ID Number					

Enter examinee's ID number here.

Graduate School Entrance Examination

Problems for Academic Year 2020

General and Special Subjects

August 20, 2019　13:30 - 16:00　(150 minutes)

Instructions

(1) Do not open this booklet until there is an instruction to start the exam.

(2) This booklet has 21 pages. Notify the proctor if you find a missing page, incorrect collating, or unclear printing.

(3) Use a black pencil for writing answers.

(4) There is a total of eight problems, covering two subjects: four problems from mathematics and four from physics. Answer Problem 1 (compulsory) and two more problems from Problems 2 through 8. You can choose two problems from one subject or two different subjects.

(5) Three answering sheets are distributed. Use one sheet for each problem. If necessary you may use the reverse side of a sheet.

(6) Write answers in English or Japanese.

(7) Enter your ID number in the specified location on the answering sheet and this booklet. Also enter the problem number on the answering sheet.

(8) Answers that contain symbols or marks that are unrelated to the problem will be considered invalid.

(9) Do not detach drafting sheets from this booklet.

(10) Do not take answering sheets or Problem booklet out of this room.

Problem 1 (Compulsory problem)

Let x, y, z, t, and k be real numbers. Answer the following questions.

(Q.1) Let $f(x, y) = x^3 + y^2 - xy$. Obtain the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.
In addition, obtain the equation of the tangent plane to the surface $z = f(x, y)$ at $(x, y, z) = (1, 2, f(1, 2))$.

(Q.2) Let $h(x) = \exp\{\exp(2x) - 1\}$. Obtain a first-order Taylor series expansion around $x = 0$ for $h(x)$. In addition, obtain the values of k and a when the limit

$$a = \lim_{x \to 0} \frac{1 - h(x)}{x^k}$$

exists and satisfies $0 < |a| < \infty$.

(Q.3) The function $\cos^{-1}$ is the inverse function of cos, and the domain and the range of $\cos^{-1}$ are $[-1, 1]$ and $[0, \pi]$, respectively. Draw the curve $y = \cos^{-1}(x + \frac{1}{2})$.
In addition, obtain the area of the region surrounded by this curve, the x-axis, and the y-axis.

(Q.4) Let $g(t)$ be a real function satisfying the differential equation

$$\frac{\mathrm{d}^2 g}{\mathrm{d}t^2} + \frac{\mathrm{d}g}{\mathrm{d}t} + \sin t = 0.$$

Obtain the general solution to this differential equation. In addition, obtain the particular solution for the initial values $g(0) = 2$ and $\dfrac{\mathrm{d}g}{\mathrm{d}t}(0) = 0$.

(Q.5) Let $D = \{\, (x, y) \,|\, 0 \le 2x - y \le 1,\ 0 \le x + 3y \le 2 \,\}$. Using the change of variables $u = 2x - y$ and $v = x + 3y$, obtain the value of the double integral

$$\iint_D \frac{(2x - y)^3}{4 + (x + 3y)^2}\, \mathrm{d}x\, \mathrm{d}y.$$

Problem 2 (Mathematics)

Consider a square matrix of order four

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and let $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^\top$ represent a four dimensional real vector, that is, $\boldsymbol{x} \in \mathbb{R}^4$. Assume that a hyperplane P includes the origin and is perpendicular to a vector $\boldsymbol{n} \in \mathbb{R}^4$ and that a hypersphere surface S of radius $a > 0$ is centered at a point $\boldsymbol{c} \in \mathbb{R}^4$, that is,

$$\begin{aligned} \text{Hyperplane } P &= \{\boldsymbol{x} \in \mathbb{R}^4 |\ \boldsymbol{n}^\top \boldsymbol{x} = 0\} \quad \text{and} \\ \text{Hypersphere surface } S &= \{\boldsymbol{x} \in \mathbb{R}^4 |\ \|\boldsymbol{x} - \boldsymbol{c}\| = a\}, \end{aligned}$$

where $\|\boldsymbol{x}\| = \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2}$, $\|\boldsymbol{n}\| = 1$, and $\top$ indicates transpose. Answer the following questions.

(Q.1) Obtain the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ of A and the corresponding eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and $\boldsymbol{e}_4$.

(Q.2) A set $Q = \{A\boldsymbol{x} |\ \boldsymbol{x} \in \mathbb{R}^4\}$ forms a linear space. Obtain the dimension and the basis vectors of Q. The basis vectors need not be orthogonal to one another.

(Q.3) The distance g between the hyperplane P and the hypersphere surface S is defined as the minimum value of $\|\boldsymbol{p} - \boldsymbol{s}\|$ for $\boldsymbol{p} \in P$ and $\boldsymbol{s} \in S$, that is,

$$g = \min_{\boldsymbol{p} \in P,\ \boldsymbol{s} \in S} \|\boldsymbol{p} - \boldsymbol{s}\|.$$

Express g in terms of $\boldsymbol{n}, \boldsymbol{c}$, and a.

(Q.4) Assume $\boldsymbol{n} = (1, 0, 0, 0)^\top$, $\boldsymbol{c} = (1, 1, 1, 1)^\top$, and $a = 1$. The distance $d(\boldsymbol{x})$ between $\boldsymbol{x}$ and $L = \{A\boldsymbol{p} |\ \boldsymbol{p} \in P\} \subset Q$ is defined as the minimum distance between $\boldsymbol{x}$ and $\boldsymbol{l} \in L$, that is,

$$d(\boldsymbol{x}) = \min_{\boldsymbol{l} \in L} \|\boldsymbol{x} - \boldsymbol{l}\|.$$

(1) Express $d(\boldsymbol{x})$ in terms of x_1, x_2, x_3, and x_4.

(2) Obtain $\max_{\boldsymbol{s} \in S} d(\boldsymbol{s})$, the maximum value of $d(\boldsymbol{s})$ for $\boldsymbol{s} \in S$.

Problem 3 (Mathematics)

Let $a > 0, x, y, X$, and Y be real numbers, e be the base of the natural logarithm, and i be the imaginary unit. Answer the following questions.

(Q.1) Show

$$\lim_{X\to\infty}\left|\int_0^Y \mathrm{e}^{-a(X+\mathrm{i}y)^2}\mathrm{d}y\right| = 0. \tag{1}$$

(Q.2) Calculate the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-a(x+\mathrm{i}Y)^2}\mathrm{d}x$$

by using contour integration along the path shown in Figure 1. You may use Eq. (1) and $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}}$.

Consider the partial differential equation of a two-variable real function $u(x,t)$ given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

where $t > 0$ is a real number and $u(x,t)$ satisfies $u(x,t) \to 0$ and $\dfrac{\partial u}{\partial x}(x,t) \to 0$ as $|x| \to \infty$.

(Q.3) The Fourier transform of $u(x,t)$ is defined as

$$U(k,t) = \int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x,$$

where k is a real number. Obtain an ordinary differential equation of U with respect to t by calculating the Fourier transform of Eq. (2).

(Q.4) Obtain $U(k,t)$ by solving the ordinary differential equation derived in (Q.3) under the condition $u(x,0) = \delta(x-1)$. Here, the delta function $\delta(x)$ is defined as

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

(Q.5) The inverse Fourier transform of $U(k,t)$ is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,t)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k.$$

Obtain $u(x,t)$ under the condition in (Q.4).

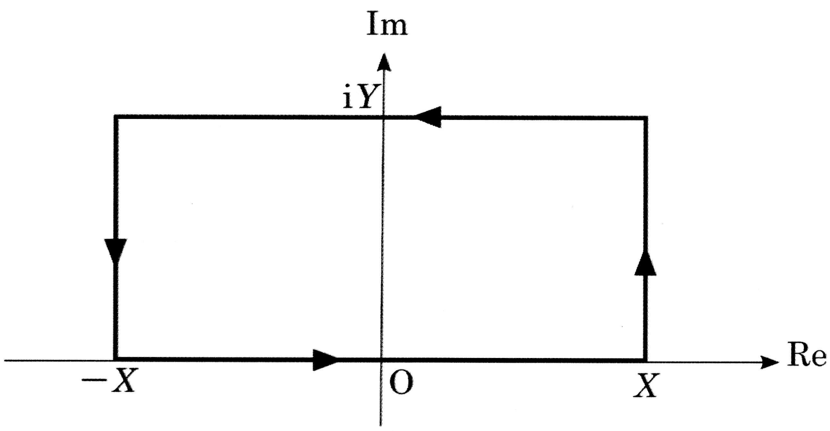

Figure 1: Integral path.

Problem 4 (Mathematics)

Consider two school routes to the Kashiwa campus shown in Tables 1 and 2. Assume that each time required for walking, bicycle, train, and bus follows a normal distribution with the mean and standard deviation shown in Tables 1 and 2, and these are independent of each other. Table 3 shows the probability that random variable X_0 following the standard normal distribution takes a value no less than x_0. Let probability density function f_X and moment generating function M_X of random variable X following the normal distribution with mean μ and standard deviation $\sigma > 0$ be

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad M_X(t) = \mathbb{E}_X[\exp(tX)],$$

where $\mathbb{E}_X$ is the expectation over random variable X, and t is a real number. Answer the following questions. You may use $\sqrt{2} \approx 1.4$, $\sqrt{3} \approx 1.7$, $\sqrt{5} \approx 2.2$, and $\sqrt{7} \approx 2.6$.

(Q.1) When $Z = \alpha X + \beta$ follows the standard normal distribution, express α and β in terms of μ and σ.

(Q.2) Random variable X_i $(i = 1, 2)$ independently follows the normal distribution with mean μ_i and standard deviation $\sigma_i > 0$. For real numbers a and b, obtain A, B, and C satisfying $M_{aX_1+bX_2}(t) = \exp(At^2 + Bt + C)$.

(Q.3) From Table 4, choose the interval containing the probability that the total time of school route 1 is no less than 75 minutes.

(Q.4) Suppose that you use school route 1. Obtain the departure time such that you arrive at the Kashiwa campus by 10 a.m. with probability 99%.

(Q.5) From Table 4, choose the interval containing the probability that the total time of school route 1 is longer than that of school route 2 by no less than 25 minutes.

Table 1: School route 1.

Routes	Mean	Standard deviation
Walking	15 min.	2 min.
↓		
Train A	25 min.	4 min.
↓		
Bus A	25 min.	5 min.

Table 2: School route 2.

Routes	Mean	Standard deviation
Bicycle	25 min.	4 min.
↓		
Train B	15 min.	8 min.
↓		
Bus B	15 min.	8 min.

Table 3: Probability.

x_0	$\Pr\{X_0 \geq x_0\}$
1.00	16%
1.28	10%
1.44	7.5%
1.65	5.0%
1.96	2.5%
2.33	1.0%
2.58	0.5%

Table 4: Choices.

Interval
10–16%
7.5–10%
5.0–7.5%
2.5–5.0%
1.0–2.5%
0.5–1.0%
0.0–0.5%

Problem 5 (Physics)

Consider a point mass with mass m which moves only in the vertical direction between a horizontal floor and a horizontal ceiling. The vertically upward axis is denoted as y-axis as shown in Figure 1. The positions of the floor and the ceiling are $y = 0$ and $y = H$, respectively. The coefficient of reflection between the point mass and the floor/ceiling is unity (i.e. perfectly elastic collision). The gravitational acceleration is g.

(Q.1) The point mass starts to fall from the position $y = H/2$ with zero initial velocity. Answer the following questions assuming that the air resistance is negligible.

(1) Obtain the amount of time T from the start of the point mass's motion till its first arrival at the floor. Also find the impulse I exerted on the floor during a collision.

(2) Assume that the point mass repeats the vertically reciprocating motion for a sufficiently long time. Find the time-averaged force exerted on the floor.

(Q.2) The point mass starts to fall from the position $y = H/2$ with initial velocity of $dy/dt = -V_0$. Answer the following questions assuming that the air resistance is negligible.

(1) Describe the requirement for the point mass to reach the ceiling after the bounce from the floor.

(2) Assume that the point mass repeats the vertically reciprocating motion for a sufficiently long time under the requirement obtained in (1). Find the time-averaged forces exerted on the floor and the ceiling, respectively.

(Q.3) Next, assume that the air resistance proportional to the velocity, $-k(dy/dt)$, exerts on the point mass. k is a positive constant. The point mass starts to fall at time $t = 0$ from a position with zero initial velocity and first arrives at the floor at time $t = m/k$. Let the base of the natural logarithm be e. Answer the following questions.

(1) Write down the equation of motion of the point mass.

(2) Obtain the velocity of the point mass when it arrives at the floor for the first time.

(3) Find the time when the point mass reaches the highest position after the first bounce from the floor.

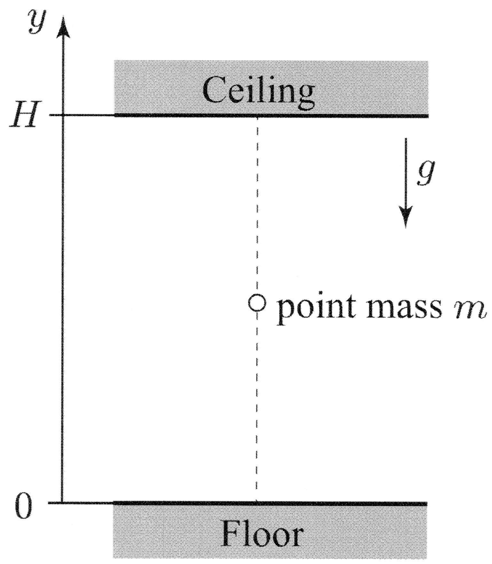

Figure 1

Problem 6 (Physics)

Consider the motion of a particle (mass m, charge q) moving at velocity $\boldsymbol{v}$ in a uniform magnetic field $\boldsymbol{B}$. The influence of gravity is ignored.

(Q.1) Express the force applied to the particle using q, $\boldsymbol{v}$, and $\boldsymbol{B}$.

(Q.2) Let $|\boldsymbol{B}| = B$ and let $v_\perp$ be the absolute value of the velocity on a plane perpendicular to the magnetic field. When $v_\perp \neq 0$ is assumed, the motion of the particle projected on this plane is circular. Then, find its radius and angular frequency.

(Q.3) There is a grounded conductive hollow cylinder with an inner radius 1.0×10^{-1}[m] with an axis in the direction of the magnetic field $\boldsymbol{B}$. From a certain point on the central axis of this cylinder, an electron with its kinetic energy W[eV] is injected in the direction perpendicular to the axis. In this case, find the minimum value of W necessary for the electron to reach the inner wall of the cylinder. Also, let $|\boldsymbol{B}| = 5.0 \times 10^{-4}$[T], the elementary charge be 1.6×10^{-19}[C], and the electron mass be 9.1×10^{-31}[kg].

There is an axis symmetric static magnetic field of which strength gradually varies in space. Consider the local cylindrical coordinate system (r, θ, z) with the z-axis as the axis of symmetry (Fig.1). Let the magnetic field be $\boldsymbol{B} = (B_r, B_\theta, B_z)$, and $B_\theta = 0$. Consider a particle (mass m, charge q) with circular motion whose central axis is the z-axis. Assume that the orbital radius of the particle is sufficiently smaller than the spatial scale of the magnetic field strength variation.

(Q.4) Express the formula regarding the divergence of the magnetic field (Gauss's law) in the above-mentioned cylindrical coordinate system. Also, express the r component of the magnetic field $\boldsymbol{B}$ using r and $\dfrac{\partial B_z}{\partial z}$. You may use the following equation of divergence in the cylindrical coordinate system. You can assume that $\dfrac{\partial B_z}{\partial z}$ is independent of r.

$$\mathrm{div}\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

(Q.5) Express the force f_z exerted on the particle along the z-axis using the magnetic moment μ_m. The magnetic moment is defined by the product IS where I is circular current surrounding the area S.

(Q.6) When the particle moves by infinitesimal distance Δz in z direction by the force f_z, find the amount of change in kinetic energy of the particle along the z-axis. Also, show that the magnetic moment μ_m is conserved.

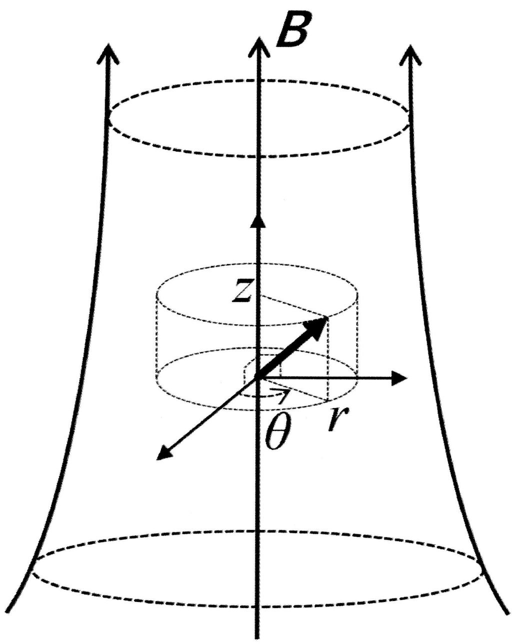

Figure 1

Problem 7 (Physics)

Consider three dimensional gas which consists of N particles with mass m in equilibrium with a heat bath of temperature T. Let the volume occupied by the gas be V. Assume there are no interactions between the particles.

(Q.1) Let the position and momentum of the i-th particle be $\boldsymbol{x}_i$ and $\boldsymbol{p}_i$. Assuming the rotation and vibration of the particles can be ignored, answer the following questions.

(1) Write down the Hamiltonian H of this system.

(2) Calculate the partition function

$$Z = \frac{1}{N!} \int \left(\prod_{i=1}^{N} \frac{\mathrm{d}\boldsymbol{p}_i \mathrm{d}\boldsymbol{x}_i}{h^3} \right) \exp(-\beta H)$$

where $\beta = 1/(k_\mathrm{B} T)$ and k_B is the Boltzmann constant, and h is the Planck constant. You may use $\int_{-\infty}^{\infty} \mathrm{d}t \exp(-at^2) = \sqrt{\pi/a}$.

(3) Calculate the Helmholtz free energy $F = -k_\mathrm{B} T \ln Z$. Apply the Stirling's formula $\ln N! \simeq N \ln N - N$ valid when $N \gg 1$, and show that F is extensive.

(4) Calculate the internal energy $U = F - T(\partial F/\partial T)$, and show that the heat capacity at constant volume is

$$C_V = \left.\frac{\partial U}{\partial T}\right|_V = \frac{3}{2} N k_\mathrm{B}.$$

(Q.2) Let the particles be diatomic molecules with moment of inertia I and an electric dipole moment q, and consider their rotational motion. When homogeneous external electric field E is applied, let the angle between the axis of the i-th molecule and the electric field be θ_i, and its azimuthal angle be ϕ_i. The contribution of the rotational motion to the Hamiltonian H_rot can be written as

$$H_\mathrm{rot} = \sum_{i=1}^{N} \left(\frac{1}{2I} \left(p_{\theta,i}^2 + \frac{p_{\phi,i}^2}{\sin^2 \theta_i} \right) - qE \cos \theta_i \right),$$

where $p_{\theta,i}$ and $p_{\phi,i}$ are the angular momentum of the i-th molecule in the θ_i and ϕ_i direction, respectively.

(1) Calculate the contribution of the rotational motion to the partition function

$$Z_{\mathrm{rot}} = \int \left(\prod_{i=1}^{N} \frac{\mathrm{d}p_{\theta,i}\mathrm{d}p_{\phi,i}\mathrm{d}\theta_i\mathrm{d}\phi_i}{h^2} \right) \exp(-\beta H_{\mathrm{rot}}).$$

(2) Calculate the contribution of the rotational motion to the internal energy. Evaluate the contributions of the rotational motion to the heat capacity at constant volume in the two limits, $k_{\mathrm{B}}T \ll qE$ and $k_{\mathrm{B}}T \gg qE$, respectively.

(3) Calculate the polarization of the system (the expectation value of $\sum_{i=1}^{N} q\cos\theta_i$).

Problem 8 (Physics)

Consider a particle with mass m and energy E whose motion along x axis under a potential energy $V(x)$ is described by a wave function $\psi(x)$. The domain of x is given as $(-\infty, \infty)$. Here, a stationary state of the particle is considered and the time-dependence of the wave function can be ignored. Probability current density is expressed as follows.

$$j(x) = -\frac{i\hbar}{2m}\left(\psi^*(x)\frac{\partial}{\partial x}\psi(x) - \psi(x)\frac{\partial}{\partial x}\psi^*(x)\right)$$

Answer the following questions. $\hbar$ is the reduced Planck constant ($\hbar = h/2\pi$) and $*$ represents a complex conjugate.

(Q.1) A wave function representing a wave propagating toward $+x$ direction is given by $\psi(x) = A\exp(ikx)$. Obtain the probability current density for this wave function. A and k are complex and real constants, respectively.

(Q.2) Obtain the probability current density for the wave function expressed by $\psi(x) = A\exp(-kx)$. A and k are complex and real constants, respectively.

(Q.3) Consider a potential energy defined as $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $0 \leq x$. A particle is injected to the potential from a region $x < 0$ toward $+x$ direction. The energy of the particle E is given by $0 < E < V_0$. Obtain a wave function for each region using the requirement of the smooth connection of the wave functions. For the region $x < 0$ the wave function can be represented as $\psi(x) = A\exp(ikx) + B\exp(-ikx)$, where A and B are complex constants. You may express answers using A.

(Q.4) On the results of (Q.3), obtain the absolute values of the probability current densities of the incident wave, the reflected wave and the transmitted wave. And calculate the reflectivity and the transmittance of this potential.

(Q.5) Consider a potential energy defined as $V(x) = 0$ for $x < 0$ and $a < x$, and $V(x) = V_0$ for $0 \leq x \leq a$. A particle with the energy E $(0 < E < V_0)$ is injected to the potential from the region $x < 0$ toward $+x$ direction. Obtain the reflectivity and the transmittance of this potential. a is a positive and real number.

(Q.6) For the results of (Q.5) and the case where $E = V_0/2$ and $V_0 \gg \hbar^2/(ma^2)$, obtain the transmittance.