Department of Complexity Science and Engineering
Graduate School of Frontier Sciences
The University of Tokyo

ID Number					

Enter examinee's ID number here.

Graduate School Entrance Examination

Problems for Academic Year 2019

General and Special Subjects

August 21, 2018 13:30 - 16:00 (150 minutes)

Instructions

(1) Do not open this booklet until there is an instruction to start the exam.

(2) This booklet has 19 pages. Notify the proctor if you find a missing page, incorrect collating, or unclear printing.

(3) Use a black pencil for writing answers.

(4) There is a total of eight (8) problems, covering two subjects: four (4) problems from mathematics and four (4) from physics. Answer Problem 1 (compulsory) and two (2) more problems from Problems 2 through 8. You can choose two (2) problems from one (1) subject or two (2) different subjects.

(5) Three (3) answering sheets are distributed. Use one (1) sheet for each problem. If necessary you may use the reverse side of a sheet.

(6) Write answers in English or Japanese.

(7) Enter your ID number in the specified location on the answering sheet and this booklet. Also enter the problem number on the answering sheet.

(8) Answers that contain symbols or marks that are unrelated to the problem will be considered invalid.

(9) Do not detach drafting sheets from this booklet.

(10) Do not take answering sheets or Problem booklet out of this room.

Problem 1 (Compulsory problem)

Let $f(x, y)$ $(x, y \in \mathbb{R},\ (x, y) \neq (0, 0))$ be a real function satifying the differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + f = 0. \tag{1}$$

Consider this differential equation in polar coordinates (r, θ) $(r, \theta \in \mathbb{R},\ r > 0)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Let $g(r, \theta) = f(x(r, \theta),\ y(r, \theta))$ and answer the following questions.

(Q.1) Express r in terms of x and y.

(Q.2) When $\dfrac{\partial f}{\partial x} = A(r, \theta)\dfrac{\partial g}{\partial r} + B(r, \theta)\dfrac{\partial g}{\partial \theta}$, obtain $A(r, \theta)$ and $B(r, \theta)$.
You may use $\dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r}$.

(Q.3) When $\dfrac{\partial^2 f}{\partial x^2} = D(r, \theta)\dfrac{\partial^2 g}{\partial r^2} + E(r, \theta)\dfrac{\partial g}{\partial r} + F(r, \theta)\dfrac{\partial^2 g}{\partial \theta^2} + G(r, \theta)\dfrac{\partial g}{\partial \theta} + H(r, \theta)\dfrac{\partial^2 g}{\partial r \partial \theta}$, obtain $D(r, \theta)$, $E(r, \theta)$, $F(r, \theta)$, $G(r, \theta)$ and $H(r, \theta)$.

(Q.4) Express a differential equation for $g(r, \theta)$ by converting the variables of the differential equation (1) to polar coordinates. You may use $\dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r}$.

(Q.5) Suppose that a solution to the differential equation obtained in (Q.4) is expressed as $g(r, \theta) = R(r)\sin\theta$. Show that $R(r)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left(1 - \frac{1}{r^2}\right) R = 0. \tag{2}$$

(Q.6) Suppose that a solution to the differential equation (2) is expressed by a termwise differentiable series $R(r) = r + \sum_{m=1}^{\infty} C_m r^{m+1}$.

(1) Obtain C_1 and C_2.
(2) Express C_{n+2} in terms of C_n for integers $n \geq 1$.

Problem 2 (Mathematics)

For integers $n \geq 1$, consider three-term recurrence relation

$$x_{n+2} = x_{n+1} + x_n, \ x_1 = 1, \ x_2 = 1.$$

Answer the following questions.

(Q.1) Obtain 2×2 real matrix A satisfying $\begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix} = A \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}$.

(Q.2) Obtain λ_+ and λ_- ($\lambda_+ > \lambda_-$) that are the eigenvalues of matrix A.

(Q.3) By using the diagonalization of matrix A, show

$$A^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_+^{n-1} - \lambda_-^{n-1} & \lambda_+^n - \lambda_-^n \\ \lambda_+^n - \lambda_-^n & \lambda_+^{n+1} - \lambda_-^{n+1} \end{pmatrix}. \tag{1}$$

You may use equation (1) in the following questions.

(Q.4) Obtain real numbers α and β satisfying $x_n = \alpha\lambda_+^n + \beta\lambda_-^n$.

(Q.5) Consider three-term recurrence relation

$$y_{n+2} = y_{n+1} + y_n, \ y_1 = a, \ y_2 = b$$

for real numbers a and b. Express y_n $(n \geq 3)$ in terms of a, b, x_{n-1}, and x_{n-2}.

(Q.6) Let $D_1 = 1$ and D_n $(n \geq 2)$ be the determinant of $n \times n$ tridiagonal matrix B_n . When

$$B_2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \ B_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 1 & \ddots & & \vdots \\ 0 & -1 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{pmatrix} \quad \text{for } n \geq 3,$$

express D_n $(n \geq 3)$ in terms of x_{n-1} and x_{n-2}.

Problem 3 (Mathematics)

The Fourier transform of a real function $f(t)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t,$$

where t and ω are real numbers, e is the base of the natural logarithm, and i is the imaginary unit. The sinc function $s(\omega)$ is defined as

$$s(\omega) = \frac{\sin(\omega)}{\omega}.$$

Answer the following questions.

(Q.1) Obtain the Fourier transform of $g(t)$ and express it by using the sinc function, where $g(t)$ is given by

$$g(t) = \begin{cases} 1, & |t| \leq \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}. \end{cases}$$

(Q.2) The convolution integral of real functions $p(t)$ and $q(t)$ is defined as

$$(p * q)(t) = \int_{-\infty}^{\infty} p(\tau)q(t-\tau)\mathrm{d}\tau.$$

Obtain $h(t) = (g * g)(t)$.

(Q.3) Obtain the Fourier transform of $k(t)$ and express it by using the sinc function, where $k(t)$ is given by

$$k(t) = \begin{cases} 1 - |t|, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}$$

(Q.4) Obtain $l(t)$ whose Fourier transform is given by $L(\omega) = \{s(\frac{\omega}{2})\}^3$.

Problem 4 (Mathematics)

Let k and n be integers larger than one. Consider random variable $X \in \{1, 2, \ldots, k\}$ such that the probability of event $X = a$ is given by

$$\Pr[X = a] = \begin{cases} 2^{-a}, & a = 1, 2, \ldots, k-1, \\ 2^{-(k-1)}, & a = k. \end{cases}$$

Let $X_1, X_2, \ldots, X_n$ be random variables that are independent of each other and follow the same distribution as X. Let $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ be the minimum value of $X_1, X_2, \ldots, X_n$. Answer the following questions.

(Q.1) Let $k = 5$. Obtain $\Pr[X \geq a]$ for $a = 1, 2, 3, 4, 5$.

(Q.2) Let $k = 5$. Obtain the expectation and the variance of random variable $Z = 2^X$.

(Q.3) Let k be an integer larger than one. Obtain $\Pr[Y_n \geq a]$ for $a = 1, 2, \ldots, k$.

(Q.4) Let k be an integer larger than one. Obtain the expectation of Y_n.

Problem 5 (Physics)

Suppose the earth is a rigid sphere rotating with a constant angular velocity. In the stationary coordinate system $\Sigma(x, y, z)$ whose origin is at the earth's center O, the angular velocity vector is expressed as $\boldsymbol{\omega} = (0, 0, \omega)$. As shown in Figure 1, let $\Sigma'(x', y', z')$ be the rotating coordinate system fixed to the earth's surface with the origin at point P on the northern hemisphere at latitude $\theta(0<\theta<\frac{\pi}{2})$, and define the directions towards the center as $-z'$ axis, towards the south as $+x'$ axis and towards the east as $+y'$ axis on the tangential plane.

(Q.1) When the position of point P is expressed by the vector $\boldsymbol{r}$ from the center of the earth, show that its time derivative $\dot{\boldsymbol{r}}$ can be expressed by Equation (1),

$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}. \tag{1}$$

(Q.2) Express the components of the angular velocity vector in Σ'.

(Q.3) A vector specifying the position of a certain point is expressed as $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$ in Σ, and as $\boldsymbol{r}' = x'\boldsymbol{e}_{x'} + y'\boldsymbol{e}_{y'} + z'\boldsymbol{e}_{z'}$ in Σ' using the basis vectors for each coordinate system. A point mass at point P is moving at velocity $\boldsymbol{v}'$ and acceleration $\boldsymbol{a}'$ in Σ'. Derive Equations (2) and (3) for velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ in Σ,

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}, \tag{2}$$

$$\boldsymbol{a} = \boldsymbol{a}' + 2\boldsymbol{\omega} \times \boldsymbol{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}). \tag{3}$$

(Q.4) In Σ', write down the equation of motion for a point mass which experiences the gravitational acceleration $\boldsymbol{g}' = (0, 0, -g')$ towards the center of the earth, using Equation (3). Terms of order ω^2 may be ignored.

(Q.5) In Σ', find the time it takes for the point mass initially located at height h' above point P with initial velocity 0 to reach the earth's surface. Also obtain the deviation of the landing point from point P in the y' direction. Here, h' is sufficiently small compared to the earth's radius, and θ and g' can be assumed to be constant. Terms of order ω^2 and the effect of air viscosity may be ignored.

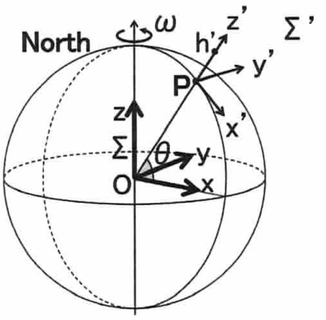

Figure 1

Problem 6 (Physics)

Consider a vacuum space with permittivity of ε_0 in Cartesian coordinates (x, y, z). A total charge of $+q$ $(q > 0)$ is uniformly distributed inside a sphere with center at the origin and radius R as shown in Figure 1. Answer the following questions under the assumption that the charge distribution inside the sphere does not change.

(Q.1) Express the magnitude of the electric field $|E|$ inside and outside the sphere as a function of the distance $r = \sqrt{x^2 + y^2 + z^2}$ from the origin.

(Q.2) Express the electrostatic potential ϕ inside and outside the sphere as a function of the distance r from the origin. Here, the electrostatic potential at infinity is taken to be zero.

Consider that a point charge $-q$ can be placed with an initial velocity of zero at an arbitrary position inside or outside the sphere in addition to the charge distribution shown in Figure 1.

(Q.3) Find the position(s) where the point charge remains stationary.

(Q.4) Consider that a spatially uniform electric field E_0 $(E_0 > 0)$ in the z direction is applied. Show the condition that there exists at least one position that the point charge remains stationary, and find the stationary position(s) when this condition is satisfied.

Next, consider the case in which a point charge $+q$ is located at the point $(x, y, z) = (0, 0, d/2)$ in vacuum and another point charge $-q$ is located at $(x, y, z) = (0, 0, -d/2)$, as shown in Figure 2. The electrostatic potential generated by these two point charges at a position far away from the origin $(r = \sqrt{x^2 + y^2 + z^2} \gg d)$ can be approximated as

$$\phi \approx \frac{qd}{4\pi\varepsilon_0} \frac{z}{r^3}.$$

(Q.5) Find the position far away from the origin where the electric field component in the xy plane is zero.

(Q.6) Find the position far away from the origin where the z component of the electric field is zero.

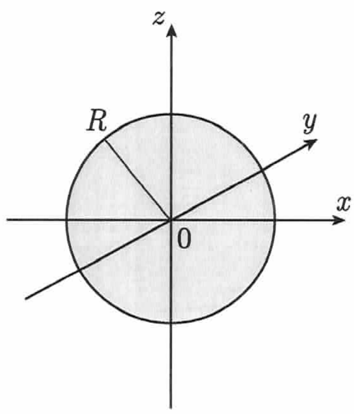

Figure 1

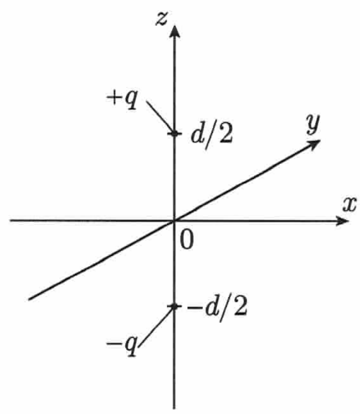

Figure 2

Problem 7 (Physics)

Let's consider phase equilibria of pure water in terms of thermodynamic variables, temperature T and pressure P. The molar entropies S and the molar volumes V of gas, liquid and solid phases of water are expressed as (S_g, V_g), (S_l, V_l) and (S_s, V_s), respectively. Assume that the molecular weight of water is 18 and the melting point of water under 1 atm is 0 °C. Water vapor can be treated as ideal gas. For simplicity, use 0 °C = 273 K and 1 atm = 0.10 MPa. You may use the following values, if necessary. The gas constant $R = 8.31\ \mathrm{JK^{-1}mol^{-1}}$, the heat of melting of ice at 0 °C and 1 atm : $6.02\ \mathrm{kJmol^{-1}}$, the heat of vaporization of water : $45.0\ \mathrm{kJmol^{-1}}$, the density of water at 0 °C and 1 atm: $1.00\ \mathrm{g/cm^3}$, the density of ice at 0 °C and 1 atm : $0.917\ \mathrm{g/cm^3}$.

(Q.1) The Gibbs energy G is defined as $G = U + PV - TS$ using the internal energy U. Show that the total derivative of the Gibbs energy is given by $\mathrm{d}G = -S\mathrm{d}T + V\mathrm{d}P$. You may use $\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V$.

(Q.2) Letting the gas phase and the liquid phase be in equilibrium, express the relationship between their respective Gibbs energies G_g and G_l.

(Q.3) Q is defined as the molar latent heat of vaporization associated with the phase transition from the liquid phase to the gas phase. Express Q in terms of tempearture and entropies.

(Q.4) Derive the expression for $\mathrm{d}P/\mathrm{d}T$, when the gas phase and the liquid phase are in equilibrium.

(Q.5) Calculate the ratio of the molar volume of the gas phase to the molar volume of the liquid phase at 0 °C and 1 atm to two significant figures.

(Q.6) Obtain the expression for P as a function of T from the result of Q.4. Assume that $V_g \gg V_l$ and V_l is neglibigle. The pressure takes the value P_0 at the reference temperature T_0. Assume that Q is independent of the temperature.

(Q.7) Calculate the value of $\mathrm{d}P/\mathrm{d}T$ around 0 °C for the phase transition between the solid phase and the liquid phase to two significant figures. The unit should be explicitly stated.

(Q.8) Figure 1 illustrates the phase diagram of water around 0 °C. Describe the words that fit in the blanks (α) - (δ). Choose the appropriate boundary line from a, b, c and explain the reason briefly.

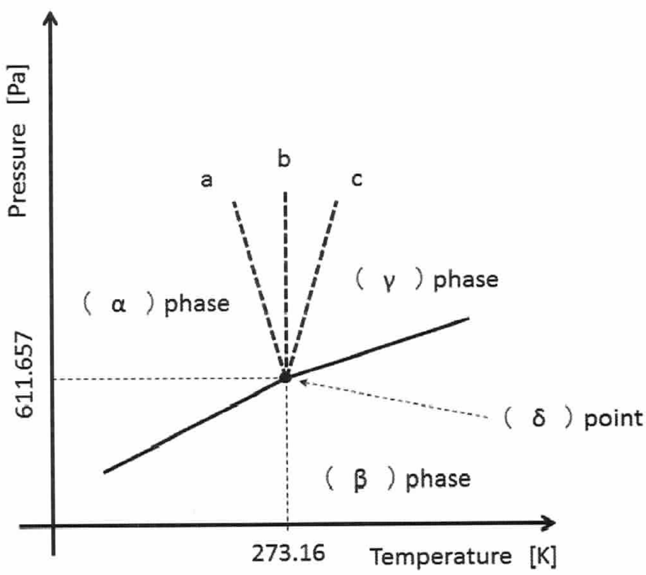
Pressure [Pa]
a
b
c
(γ) phase
(α) phase
611.657
(δ) point
(β) phase
273.16
Temperature [K]

Figure 1

Problem 8 (Physics)

Consider a one-dimensional quantum mechanical harmonic oscillator described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2},$$

where m is the mass of the point mass, ω is a parameter with the dimension of the angular frequency, $\hat{x}$ is the position operator and $\hat{p}$ is the momentum operator. The creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ are defined as

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{\mathrm{i}}{m\omega}\hat{p}\right), \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{\mathrm{i}}{m\omega}\hat{p}\right).$$

Here, i is the imaginary unit and $\hbar$ is the Planck's constant h divided by 2π. The commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ may be used as necessary. The expected value of a physical quantity A expressed by the operator $\hat{A}$ is written as $\langle\hat{A}\rangle$, and the standard deviation, $\Delta A \equiv \sqrt{\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle}$.

(Q.1) Express the position operator $\hat{x}$ and the momentum operator $\hat{p}$ in terms of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$.

(Q.2) The ground state $|0\rangle$ satisfies $\hat{a}|0\rangle = 0$. Answer the following questions on the ground state.

(1) Find the expectation values $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{p}^2\rangle$.

(2) Find the product of the standard deviations of the position and the momentum $(\Delta x \Delta p)$.

(Q.3) Rewrite the Hamiltonian $\hat{H}$ in terms of the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$.

(Q.4) Express the commutation relation $[\hat{a}, (\hat{a}^\dagger)^n]$ (n is a natural number) in terms of $\hat{a}^\dagger$ and n.

(Q.5) The n-th excited state (n is a natural number) can be written as

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle.$$

Show that $|n\rangle$ is an eigenstate of the number operator $\hat{n}$, and find the eigenvalue of $|n\rangle$ for the Hamiltonian $\hat{H}$.

(Q.6) Consider a state expressed in terms of a complex number α

$$|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{\alpha \hat{a}^\dagger} |0\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} (\hat{a}^\dagger)^n |0\rangle .$$

Answer the following questions on this state. e is the base of the natural logarithm.

(1) Show that this state is an eigenstate of the annihilation operator $\hat{a}$. Find the corresponding eigenvalue.

(2) Obtain the expectation value $\langle \hat{n} \rangle$ and the standard deviation Δn of the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$.