Department of Complexity Science and Engineering
Graduate School of Frontier Sciences
The University of Tokyo

ID Number					

Enter examinee's ID number here.

Graduate School Entrance Examination

Problems for Academic Year 2018

General and Special Subjects

August 22, 2017 13:30 - 16:00 (150 minutes)

Instructions

(1) Do not open this booklet until there is an instruction to start the exam.
(2) This booklet has 17 pages. Notify the proctor if you find a missing page, incorrect collating, or unclear printing.
(3) Use a black pencil for writing answers.
(4) There is a total of eight (8) problems, covering two subjects: four (4) problems from mathematics and four (4) from physics. Answer Problem 1 (compulsory) and two (2) more problems from Problems 2 through 8. You can choose two (2) problems from one (1) subject or two (2) different subjects.
(5) Three (3) answering sheets are distributed. Use one (1) sheet for each problem. If necessary you may use the reverse side of a sheet.
(6) Write answers in English or Japanese.
(7) Enter your ID number in the specified location on the answering sheet and this booklet. Also enter the problem number on the answering sheet.
(8) Answers that contain symbols or marks that are unrelated to the problem will be considered invalid.
(9) Do not detach drafting sheets from this booklet.
(10) Do not take answering sheets or Problem booklet out of this room.

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Problem 1 (Compulsory problem)

Answer the following questions for the function $f(x, y) = (x + y)\mathrm{e}^{-(x^2+y^2)}$, where x and y are real numbers and e is the base of the natural logarithm.

(Q.1) Obtain $\dfrac{\partial f}{\partial x}$.

(Q.2) Obtain $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$.

(Q.3) Find the second order Taylor expansion of f around the point $(x, y) = (1, -1)$.

(Q.4) Show that f has a local maximum value at $(x, y) = (0.5, 0.5)$.

(Q.5) Find the extremal value of f, where $x^2 + y^2 = 1$ and $x, y \geq 0$.

Problem 2 (Mathematics)

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$ be the sets of real numbers and non-negative real numbers, respectively. Consider three-dimensional vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \in \mathbb{R}^3$ and 3×4 matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 4}$ given by

$$\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4) = \begin{pmatrix} 5 & 1 & 4 & 2 \\ 3 & 0 & 5 & 1 \\ 1 & 4 & 0 & 3 \end{pmatrix}.$$

A four-dimensional vector is denoted by $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4$, where $\top$ represents the transpose of a vector. Let P be the set given by

$$P = \{\boldsymbol{x} \in \mathbb{R}^4_+ : x_1 + x_2 + x_3 + x_4 = 1\}.$$

Answer the following questions.

(Q.1) Let $S \subset \mathbb{R}^3$ be the set of points $\boldsymbol{Ax}$ for $\boldsymbol{x} \in \mathbb{R}^4$ satisfying $x_1 + x_3 = 1$ and $x_2 + x_4 = 1$, that is,

$$S = \left\{\boldsymbol{Ax} : \boldsymbol{x} \in \mathbb{R}^4,\ x_1 + x_3 = 1,\ x_2 + x_4 = 1\right\}.$$

Obtain unit vector $\boldsymbol{c} \in \mathbb{R}^3$ that is perpendicular to S.

(Q.2) Obtain $\boldsymbol{x} \in \mathbb{R}^4$ satisfying $x_4 = 0$ and $\boldsymbol{Ax} = \boldsymbol{a}_4$.

(Q.3) Find one of $\boldsymbol{x} \in P$ satisfying the following two conditions simultaneously. You may use the result of (Q.2).

(a) $x_4 = 0$,

(b) $\boldsymbol{y}\boldsymbol{a}_4 \leq \boldsymbol{y}\boldsymbol{Ax}$ for any $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_+$.

(Q.4) Show that for any fixed $\boldsymbol{x}' \in P$ there exists $\boldsymbol{x} \in P$ satisfying the following two conditions simultaneously. You may use the result of (Q.3).

(a) $x_4 = 0$,

(b) $\boldsymbol{y}\boldsymbol{Ax}' \leq \boldsymbol{y}\boldsymbol{Ax}$ for any $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_+$.

Problem 3 (Mathematics)

The Fourier transform of function $f(t)$ and its inverse Fourier transform are defined as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t,$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega,$$

where t and ω are real numbers, e is the base of the natural logarithm, and i is the imaginary unit. Answer the following questions. You may use the delta function $\delta(\omega)$ given by

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

(Q.1) Obtain the Fourier transform of $\cos(\omega_0 t)$, where ω_0 is a real constant.

(Q.2) Show that $F(\omega)$ is a real function when $f(t)$ is a real even function.

(Q.3) Consider a real even function $f(t)$ given by

$$f(t) = \sum_{n=0}^{N} \cos(\omega_n t),$$

where N is a positive integer. When ω_n is given by

$$\omega_n = \frac{(1+4n)\pi}{2t_0} \qquad (n = 0, 1, 2, \ldots),$$

where t_0 is a positive real constant, show that $f(t)$ satisfies

$$f(t - t_0) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(\omega)\sin(\omega t)\mathrm{d}\omega.$$

(Q.4) Consider $G(\omega) = F(\omega)H(\omega)$, where $H(\omega)$ is given by

$$H(\omega) = \begin{cases} 2, & \omega > 0 \\ 1, & \omega = 0 \\ 0, & \omega < 0. \end{cases}$$

Express $g(t)$ with $f(t)$ and $f(t - t_0)$, where $g(t)$ is the inverse Fourier transform of $G(\omega)$ and $f(t)$ satisfies the equation of (Q.3).

Problem 4 (Mathematics)

Player plays a game with Machine only once. Let p_i be the probability that Player outputs i for $i = 1, 2, \ldots, n$, where $\sum_{i=1}^{n} p_i = 1$. Let q_j be the probability that Machine outputs j for $j = 1, 2, \ldots, n$, where $\sum_{j=1}^{n} q_j = 1$. Each of Player and Machine outputs an integer from 1 to n according to the probability distribution. If Player and Machine output the same integer, it is judged that Player wins. Answer the following questions.

(Q.1) Suppose that $p_i = 1/n$ for $i = 1, 2, \ldots, n$. Obtain the probability that Player wins.

(Q.2) Suppose that $p_i = p_1\alpha^{i-1}$ for $i = 1, 2, \ldots, n$ and $q_j = q_1\beta^{j-1}$ for $j = 1, 2, \ldots, n$. Express the probability that Player wins only with α, β and n.

(Q.3) Suppose that $p_i = q_i$ for $i = 1, 2, \ldots, n$.

- **(a)** Obtain the minimum value of the probability that Player wins.
- **(b)** Suppose that the expectation of Machine's output is $(n+1)/2$. Obtain the minimum value of the probability that Player wins.

(Q.4) We call $(p_1, p_2, \ldots, p_n)$ Player's strategy. We consider the following two strategies.

- Strategy E: $(p_1, p_2, \ldots, p_{n-1}, p_n) = (0, 0, \ldots, 0, 1)$,
- Strategy R: $(p_1, p_2, \ldots, p_n) = (1/n, 1/n, \ldots, 1/n)$,

Suppose that q_n is the largest value of $q_1, q_2, \ldots, q_n$.

- **(a)** Show that the strategy E is superior to the strategy R. We say that for strategies A and B, the strategy A is superior to the strategy B if the probability that Player with the strategy A wins is higher than or equal to the probability that Player with the strategy B wins.
- **(b)** Show that the strategy E is superior to any strategies.

Problem 5 (Physics)

A point mass is attached to the end of a rod of a pendulum as shown in Figure 1. The length of the rod can change with time. Let the mass of the point mass be m, the gravitational acceleration be g, the length of the rod be l, and the swing angle measured from the lowest position be θ. The mass of the rod, the friction at the pivot, and the air resistance can be neglected. Assume that the absolute value of θ is always smaller than 90°.

Initially the rod has a length of l_0 and is supported at rest at Point A where the swing angle measured from the bottom is θ_0 (< 0). The rod is then released gently. When the point mass reaches the lowest position (Point B), the rod length is shortened to l_1 in a short time and the point mass is raised to Point C. The point mass continues to move, and when the point mass reaches the highest position (Point D), the rod length is restored to l_0 in a short time and the point mass is lowered to Point E. The point mass then moves back toward Point B. Here we assume that the duration of the length change of the rod is short enough and that the change of the swing angle during this change can be neglected.

(Q.1) Obtain the velocity of the point mass v_B when it first reaches Point B, using g, l_0 and θ_0.

(Q.2) Obtain the velocity of the point mass v_C at Point C based on the conservation of angular momentum, using l_0, l_1 and v_B.

(Q.3) Obtain the ratio of the kinetic energy when the point mass returns to Point B after passing Point E relative to the kinetic energy when the point mass first reaches Point B, using l_0 and l_1.

(Q.4) Write down the equation for the second-order time derivative of θ (angular acceleration) for a case where l changes with time t arbitrarily. Based on this equation, explain that the angular acceleration in the direction of motion of the point mass occurs at the lowest position if the time derivative of l is negative there, and that the time derivative of l does not affect the angular acceleration at the highest position.

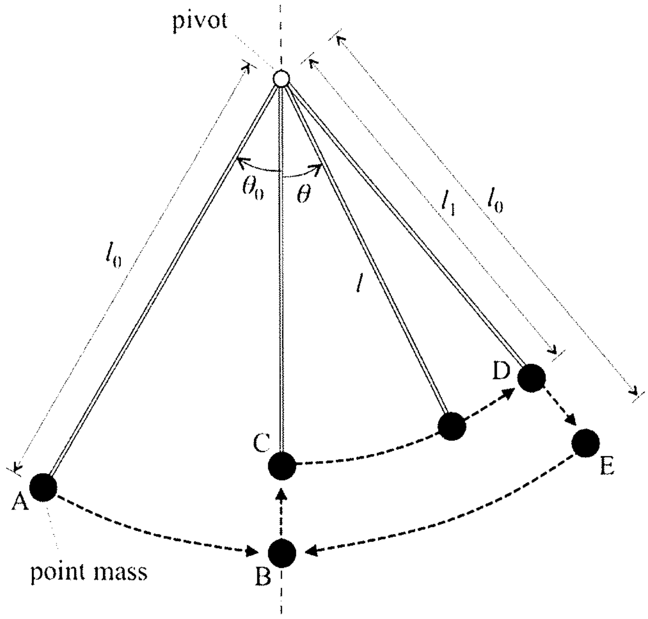


Figure 1

Problem 6 (Physics)

Answer the following questions.

(Q.1) Consider an electric circuit consisting of a resistor, a battery, an ammeter (with internal resistance $0.10\,\Omega$), and a voltmeter (with internal resistance $1.0\times10\,\Omega$), as shown in Figure 1. The internal resistance of the battery can be neglected. The ammeter and the voltmeter indicated 1.0 A and 1.0×10 V, respectively. Now, the circuit is rewired as shown in Figure 2. Answer the values indicated by the ammeter and the voltmeter to two significant figures.

(Q.2) A metal wire has a length of 2.0 m and a cross sectional area of $3.0\,\mathrm{mm}^2$. Calculate the electrical resistance of the wire to two significant figures, when the volume resistivity of the metal is $6.3\times10^{-8}\,\Omega\,\mathrm{m}$.

(Q.3) Consider an infinitely long coaxial hollow conducting cylinder with inner radius a and outer radius $3a$. Figure 3 shows a coordinate system with the z axis along the central axis of the cylinder. The cylinder carries a uniform current (with current density J) in the positive z direction. Answer the magnetic field $\vec{H}$ at the points A $(0.5a, 0, 0)$, B $(0, 2a, 0)$, and C $(3a, 3a, 0)$ in component form.

(Q.4) An infinitely long conducting cylinder with radius a and resistance R per unit length carries a uniform current I. Show the direction and the magnitude of the Poynting vector on the surface of the cylinder, and answer the total energy going into or out from the cylinder per unit length per unit time.

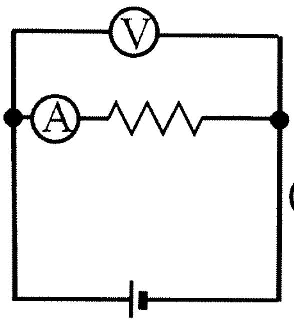


Figure 1

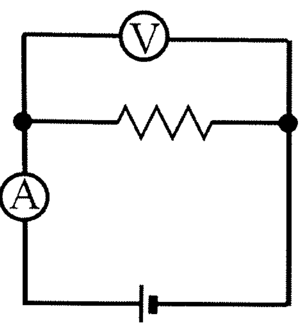
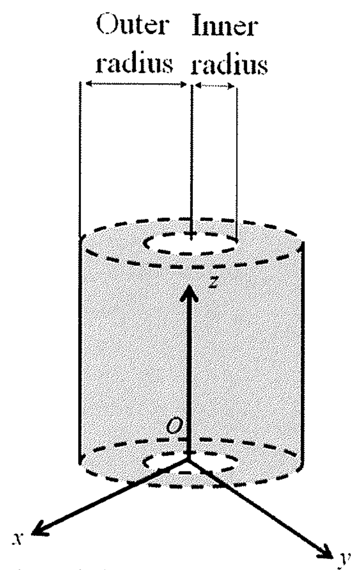


Figure 2



Coaxial hollow cylinder

Figure 3

Problem 7 (Physics)

Answer the following questions on ideal gases. Here, R is the gas constant and C_V is the molar heat capacity at constant volume. Assume that C_V is a constant.

(Q.1) Consider the entropy S of 1 mole of ideal gas with absolute temperature T, volume V and pressure P. A change in the internal energy U during a quasi-static process can be written as $dU = TdS - PdV$. Using this equation, derive the formula

$$S = C_V \log_e \frac{T}{T_0} + R \log_e \frac{V}{V_0} + S_0 \ .$$

Here, S_0 is the entropy at temperature T_0 and volume V_0.

(Q.2) Consider a system consisting of 1 mole of ideal gas A with volume V, temperature T_{A} and entropy S_{A} and 1 mole of ideal gas B with volume V, temperature T_{B} ($\neq T_{\mathrm{A}}$) and entropy S_{B}. This system is thermally isolated from the outside world. Gas A and gas B are in contact with each other across a thermally insulating wall. At a certain time, the thermally insulating wall between gas A and gas B is replaced by a thermally conducting wall. Then, their temperatures start to vary. Let T'_{A}, T'_{B} and S'_{A}, S'_{B} be the temperature and entropy of each gas after some time. Consider the entropy increment ΔS of this system, where ΔS is defined as $\Delta S = (S'_{\mathrm{A}} + S'_{\mathrm{B}}) - (S_{\mathrm{A}} + S_{\mathrm{B}})$. Assume that the state variations of gases A and B are quasi-static, and that they have the same molar heat capacity at constant volume C_V. The heat capacities of the thermally insulating wall and the thermally conducting wall can be neglected.

(1) Express ΔS in terms of C_V, T_{A}, T_{B}, T'_{A} and T'_{B}.

(2) Using the fact that the total internal energy of the entire system is conserved, express the solution of (Q.2)(1) in terms of C_V, T_{A}, T_{B} and T'_{A}.

(3) The equilibrium state of an isolated system is the state of maximum entropy. Find the condition for ΔS derived in (Q.2)(2) to have an extremum with respect to T'_{A}. Express T'_{A} and T'_{B} when this system is in equilibrium state in terms of T_{A} and T_{B}. Furthermore, prove that ΔS is always positive.

(4) When the initial temperatures of gases A and B are T_{A} and T_{B}, respectively, ΔS after this system reached equilibrium is expressed as $\Delta S_{T_{\mathrm{A}}+T_{\mathrm{B}}}$. Write down the relationship between the magnitudes of $\Delta S_{100\,\mathrm{K}+200\,\mathrm{K}}$ and $\Delta S_{50\,\mathrm{K}+250\,\mathrm{K}}$, and explain the reason.

Problem 8 (Physics)

Answer the following questions. Planck constant h divided by 2π is denoted $\hbar$, c is the speed of light in vacuum, and m is the electron mass.

(Q.1) Consider an electron bound in a one-dimensional potential well

$$V(x) = \begin{cases} 0 & (0 \le x \le L) \\ +\infty & (x < 0,\ x > L). \end{cases}$$

Find the wave functions $\psi_n(x)$ and eigenenergies E_n ($n = 1, 2, 3, ...$) of energy eigenstates. $\psi_n(x)$ should be normalized.

(Q.2) Consider a particle (mass M) bound in a harmonic oscillator potential. The Hamiltonian H of the system can be expressed as $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$, where ω is the angular eigenfrequency, and $a = \sqrt{\frac{M\omega}{2\hbar}}\left(x + \frac{\hbar}{M\omega}\frac{\partial}{\partial x}\right)$ and $a^\dagger = \sqrt{\frac{M\omega}{2\hbar}}\left(x - \frac{\hbar}{M\omega}\frac{\partial}{\partial x}\right)$ are operators.

(1) Calculate the value taken by the commutator $[a, a^\dagger] = aa^\dagger - a^\dagger a$.

(2) The ground state $|0\rangle$ satisfies $a\,|0\rangle = 0$. Compute the energy difference ΔE_{vib} between the first excited state (the excited eigenstate with the lowest excitation energy from the ground state) $a^\dagger\,|0\rangle$ and the ground state $|0\rangle$.

(Q.3) The green fluorescent protein (GFP) absorbs blue light with peak wavelength 488 nm and emits green light with peak wavelength 508 nm. Consider the fluorescence phenomenon based on electronically excited states of GFP. If necessary, you may use $hc = 1240\,\text{eV nm}$ and $\frac{\hbar^2\pi^2}{2m} = 0.373\,\text{eV nm}^2$.

(1) Compute the photon energies of electromagnetic waves with wavelengths 488 nm and 508 nm in eV unit to three significant figures.

(2) The electronically excited states of GFP are modelled by 13 electrons bound in the one-dimensional potential well considered in (Q.1) with length $L = 1.4\,\text{nm}$. Ignoring interactions between electrons, the transition energy from the ground state to the first excited state is given as $\Delta E_{\text{elec}} = E_7 - E_6$. Calculate ΔE_{elec} in eV unit to two significant figures.

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